Angular Motion of a Re-Entering Symmetric Missile

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For the case of cubic static-moment constant air density and no damping, the exact solution for the combined pitching and yawing motion of a symmetric missile can be expressed in terms of elliptic functions. A perturbation method that makes use of this exact solution is then developed to determine the effect of air-density variations and nonlinear aerodynamic Magnus and damping moments when the static moment is "strongly" nonlinear. Stability boundaries for initial conditions are computed, and the conditions for circular limit motions are derived. These circular limit motions have been experimentally observed and conditions for a possible nonlinear moment expansion have already been derived for a "slightly" nonlinear static moment by a quasilinear analysis. The various predictions of the approximate theory have been verified by numerical integration of the exact differential equations of motion. The great value of the methods developed lies in their ability to yield quickly the effect of various parameters and initial conditions on missile motion, thereby releasing the designer from the necessity of performing a large number of numerical integrations. The results of the theory, however, are limited to altitudes for which the density variations over a cycle of the motion is small (i.e., less than 10%).

Nomenclature

A = defined by Eq. (6.3)
A_{2n}, B_{2n} = defined by Eq. (6.2) and Table 1
C_1 = twice the total energy of the missile pitching and yawing motion
C_2 = twice the flight path component of the pitching and yawing angular momentum
C_D = drag coefficient
H = $(\rho Sl/2m)[\gamma a_L - C_D - k_t^{-2} d]$ [damping coefficient of Eq. (2.10)]
H_0, H_2 = coefficients in $H = H_0 + H_2\delta^2$
I_x = axial moment of inertia
I_t = transverse moment of inertia
K_j = amplitude of j mode of solution of linear equation [Eqs. (3.1–3.3)]
M = $\gamma(\rho Sl/2m)[k_t^{-2}\, c - \tilde{\sigma} a_L]$ [frequency coefficient of Eq. (2.10)]
$\hat{M}$ = $M - P^2/4$
M_0, $\hat{M}_0$ values of M and $\hat{M}$ for $\delta^2 = 0$
M^* = $\gamma(\rho Sl/2m)[k_t^{-2}\; c^* - (da_L/d\delta^2)(\delta^2)']$ [nonlinear damping coefficient of Eq. (2.10)]
M_s = $\gamma(\rho Sl/2m)[k_t^{-2}\; c_s + Pa_L]$ [side moment term of Eq. (2.10)]
P = $(I_x/I_t)(pl/V)$, gyroscopic spin
S = reference area
V = velocity
X, Y, Z = nonspinning coordinates with X along missile's symmetry axis
a = normal or static force coefficient [Eq. (2.5)]
a_L = $\gamma^{-1}(a - C_D)$, lift force coefficient
c = static moment coefficient [Eq. (2.6)]
c^* = nonlinear amplitude damping coefficient [Eq. (2.6)]
c_s = side moment coefficient [Eq. (2.6)]
d = damping moment coefficient [Eq. (2.6)]
k = modulus of elliptic functions [Eqs. (3.13–3.15)]
k_p = modulus for planar motion
k_t = $(I_t/ml^2)^{1/2}$, transverse radius of gyration
l = reference length
m = mass
m_2 = term in expansion of $\hat{M} = \hat{M}_0(1 + m_2\delta^2)$
m_c = $m_2\delta^2$ for circular motion
m_p = $4m_2(x_1x_2)^{1/2}$ for planar motion
p, $\tilde{q}$, $\tilde{r}$ = angular velocity components along X, $\tilde{Y}$, $\tilde{Z}$
s = $\int_{t_0}^{t}\left(\frac{V}{l}\right)dt$, dimensionless distance along flight path
s_g = $P^2/4M_0$, gyroscopic stability factor
x_j = generalized modal amplitudes [Eqs. (3.9) and (3.10)]
$\tilde{v}$, $\tilde{w}$ = $\tilde{Y}$ and $\tilde{Z}$ components of velocity
α_{max}, α_{av} = maximum and average angles of attack
γ = cosine of angle between missile and trajectory
δ = sine of angle between missile and trajectory
ϵ_j = perturbation of x_j [Eq. (6.1)]
$\tilde{\theta}$, $\hat{\theta}$ = argument $\tilde{\xi}$, argument $\hat{\xi}$
$\tilde{\sigma}$ = ρ'/ρ
ρ = air density
ϕ_j = j modal phase angle of linear solution [Eqs. (3.1–3.3)]
$\omega\ \tilde{\omega}$ = defined by Eqs. (3.13–3.15)
$(\ \ )'$ = primes denote derivatives with respect to s

1. Introduction

THE influence of an exponentially varying air density on the planar pitching motion of a re-entering missile with linear aerodynamic coefficients has been studied by a number of authors.[1,2] This work has been extended to the combined pitching and yawing motion of a spinning missile by Leon[3] for no aerodynamic moment and by Garber[4] for linear aerodynamic moment.

The angular motion of a symmetric missile flying in a constant-density atmosphere has been studied by a quasi-linear technique.[5,6] The quasi-linear approach has also been used for a cubic static moment, no aerodynamic damping, and an exponential atmosphere by Coakley, Laitone, and Maas.[7] This quasi-linear work was extended to nonlinear aerodynamic moments.[8]

The basic assumption of the quasi-linear method is that the actual nonlinear motion can be reasonably well approximated by a solution of the same form as that of the linearized problem. In our case, this means that, over short portions of a missile's trajectory, its angles of attack and sideslip may be represented by linear combinations of two damped sine waves that differ in phase. This assumption excludes consideration of the effect of a strongly nonlinear static moment since, even for zero damping, the solution involves the use of elliptic integrals. The classical perturbation techniques of astronomy do not have this fault and have been applied in a quite general manner by Thomas[9] and Reed[10] to the motion

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of a spinning shell flying in a constant-density atmosphere.

In this paper, we will develop this perturbation method in a much simpler fashion, but with the inclusion of the possibility of varying air density. Generalized amplitude planes will be introduced and studied for some special types of nonlinearities. The resulting predictions of missile motion will then be compared with corresponding predictions of the quasi-linear theory and with actual exact calculations.

2. Equations of Motion

We will make use of Cartesian axes that are constrained so that the X axis is aligned with the missile's axis of symmetry and the angular velocity of the axis system about this axis is zero (nonspinning coordinates). The angular orientation of the missile's axis with respect to its flight path can be determined by the $\tilde{Y}$ and $\tilde{Z}$ components† of its velocity vector $\tilde{v}$ and $\tilde{w}$. These may be combined to form the real and imaginary parts of a complex angle of attack

$$\tilde{\xi} = (\tilde{v} + i\tilde{w})/V = \delta e^{i\tilde{\theta}} \tag{2.1}$$

where δ is the sine of the angle between the missile's axis and the trajectory. In a similar way, the $\tilde{Y}$ and $\tilde{Z}$ components of the missile's angular velocity $\tilde{q}$ and $\tilde{r}$ may be combined to yield a complex dimensionless transverse angular velocity:

$$\tilde{\mu} = [(\tilde{q} + i\tilde{r})l]/V \tag{2.2}$$

If we neglect aerodynamic forces and gravity, $\tilde{\mu}$ and $\tilde{\xi}$ are related by

$$\tilde{\xi}' = i\gamma\tilde{\mu} \tag{2.3}$$

where $\gamma = (1 - \delta^2)^{1/2}$ cosine of angle of attack between missile's axis and trajectory. Even in the presence of these forces, Eq. (2.3) is a good approximation.

The use of these variables is especially appropriate when the aerodynamic force and moment acting on the missile are functionally of the same form as that for a body of revolution. We will consider a fairly general nonlinear form for the force and moment to be studied which has this property:

$$\text{drag} = \tfrac{1}{2}\rho V^2 S C_D \tag{2.4}$$

$$F_{\tilde{Y}} + iF_{\tilde{Z}} = -\tfrac{1}{2}\rho V^2 S a \tilde{\xi} \tag{2.5}$$

$$M_{\tilde{Y}} + iM_{\tilde{Z}} = -(i/2)\rho V^2 S l[(c + c^* + ic_s)\tilde{\xi} + d\tilde{\xi}'] \tag{2.6}$$

where C_D, a, and c are functions of δ^2; c^*, c_s, and d are functions of δ^2 and $(\delta^2)'$; $c^* = 0$ for $(\delta^2)' = 0$; primes denote derivatives with respect to dimensionless arc length

$$s = \int_{t_0}^{t} (V/l)dt$$

$a\tilde{\xi}$ and $ic\tilde{\xi}$ are functions of angle only; and a and c are called static-force and -moment coefficients. For linear force and moment, they are constants with

$$a = C_{N_\alpha} \qquad c = C_{M_\alpha} \tag{2.7}$$

$ic^*\tilde{\xi}$ is a damping moment that causes rotation about the same axis as the static moment, i.e., one normal to the plane of the angle of attack; c^* for a linear expansion is zero; and $c_s\tilde{\xi}$ is a moment at right angles to the static moment. Therefore c_s can be identified as a side moment coefficient. Its linear form is the Magnus moment coefficient multiplied by spin:

$$c_s = (pl/V)C_{M_{p\alpha}} \tag{2.8}$$

Finally, $id\tilde{\xi}'$ is a damping moment that opposes or encourages the angular velocity $\tilde{\xi}'$ and, for the approximation (2.3), d has the linear form:

$$d = C_{M_q} + C_{M_{\dot{\alpha}}} \tag{2.9}$$

The aerodynamic force terms corresponding to c^*, c_s, and d are not included in Eq. (2.5) since they seldom affect the angular motion. If we neglect gravity, these force and moment expansions may be combined in the usual way[11] to yield a second-order complex equation for the pitching and yawing motion of a symmetric missile:

$$\tilde{\xi}'' + [H - (\gamma'/\gamma) - iP]\tilde{\xi}' - (M + M^* + iM_s)\tilde{\xi} = 0 \tag{2.10}$$

For constant roll rate, the use of the transformation $\tilde{\xi} = \hat{\xi}e^{i(1/2)Ps}$, which makes use of a rotating coordinate system with roll rate $P/2$, allows us to write Eq. (2.10) in the simpler form

$$\hat{\xi}'' + [H - (\gamma'/\gamma)]\hat{\xi}' - (\hat{M} + M^* + i\{M_s - (P/2)[H - (\gamma'/\gamma)]\})\hat{\xi} = 0 \tag{2.11}$$

where $\hat{M} = M - (P^2/4)$.

Equation (2.11) for the combined pitching and yawing motion is a reasonably simple second-order equation in the complex variable $\hat{\xi}$, but does allow the consideration of quite general aerodynamic moment terms. To linearize this equation we would assume small angles ($\gamma' \doteq 0$), constant values of H, $\hat{M}$, and M_s and a vanishing $M^*(c^* = 0)$. The resulting linear equation would show that the basic effect of a negative $\hat{M}$ is to make the motion periodic with period $2\pi[-\hat{M}]^{-1/2}$. The $H\hat{\xi}'$ term would cause simple exponential damping whereas the complex part of the coefficient of $\hat{\xi}$ couples the pitching and yawing motion and introduces further damping or undamping of the motion.

The theory of this paper is restricted to a quadratic $\hat{M} = \hat{M}_0(1 + m_2\delta^2)$, but applies to cases where H, M^*, and M_s are arbitrary functions of δ^2 and $(\delta^2)'$. In the examples, we will only consider a quadratic $H = H_0 + H_2\delta^2$. For positive H_0 and negative H_2, this would correspond to aerodynamic damping at small angles and negative damping at large angles. This dependence of damping on amplitude has been observed in a number of wind tunnels[17] by forcing models to perform planar oscillation and is a generalization of the damping assumed for the classical van der Pol oscillator.[16] The M^* term will either be zero or proportional to $(\delta^2)'$. This term introduces a nonlinear damping, which causes a rotation in the direction of the angle, not in the direction of the angular velocity. For planar motion there is no difference, but for elliptical motion this nonlinearity induces a damping that is dependent on the shape of the angular motion.[14] Such a nonlinearity is required to explain the limit circular motions that have been observed.[6] Theoretical values of H_2 and $M_2{}^*$ can be computed from an extension of slender body theory; this has been done in Ref. 6.

The side moment term M_s is much more complex since its linearized form is the Magnus moment associated with spinning missiles. Although our examples are restricted to nonspinning symmetric missiles by a desire for algebraic simplicity, side moments need not vanish. The example considered is a side moment coefficient that vanishes for planar motion and has constant magnitude otherwise. Its algebraic sign for nonplanar motion is determined by the direction of that motion (i.e., clockwise or counterclockwise). This moment is a possible mathematical idealization of the actual side moment acting on a flared body whose flare is largely immersed in separated flow.

3. Solution for Cubic Static Moment

If only a linear static moment is considered, Eq. (2.11) becomes

$$\hat{\xi}'' - \hat{M}_0\hat{\xi} = 0 \tag{3.1}$$

† Since the conventional XYZ axes are missile fixed, we will use tilde superscripts to emphasize that these axes are not necessarily missile fixed.

For a gyroscopically stable missile, $\hat{M}_0$ is negative,

$$1/s_g \equiv (4M_0/P^2) < 1 \tag{3.2}$$

where s_g is the usual gyroscopic stability factor and Eq. (3.1) has an ellipse as its solution:

$$\hat{\xi} = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} \tag{3.3}$$

where $\phi_j = \phi_{j0} \pm [-\hat{M}_0]^{1/2}s$. The sum of the modal amplitudes $K_1 + K_2$ is the maximum angle of attack, and the difference $|K_1 - K_2|$ is the minimum angle of attack. The quasi-linear method assumes Eq. (3.3) to be a good approximate solution and averages the damping over a cycle of the motion to determine variations in the modal amplitudes. Results of this method appear as trajectories in an "amplitude" plane with K_1^2 and K_2^2 axes.

Equation (2.11) for a cubic static moment ($c = c_0 + c_2\delta^2$) assumes a very simple form

$$\hat{\xi}'' - \hat{M}_0(1 + m_2\delta^2)\hat{\xi} = 0 \tag{3.4}$$

This conservative equation can be manipulated to yield two constants of the motion. The energy equation can be obtained by multiplying Eq. (3.4) by the conjugate of $\hat{\xi}'$ and adding the result to its own conjugate

$$C_1' = 0 \tag{3.5}$$

where $C_1 = (\delta')^2 + (\delta\hat{\theta}')^2 - \hat{M}_0(\delta^2 + m_2\delta^4/2)$ and $\hat{\xi} = \delta e^{i\hat{\theta}}$. Next we multiply Eq. (3.4) by the conjugate of $\hat{\xi}$ and subtract the result from its own conjugate to obtain the equation for the flight-path component of the angular momentum of the pitching and yawing motion:

$$iC_2' = 0 \tag{3.6}$$

where $C_2 = 2\delta^2\hat{\theta}'$. Since Eqs. (3.5) and (3.6) state that the C_i are constants of the motion, this can be used to solve Eq. (3.4) by two quadratures

$$(\delta^2)' = \pm[-C_2^2 + 4C_1\delta^2 + 2\hat{M}_0(2\delta^4 + m_2\delta^6)]^{1/2} \tag{3.7}$$

$$\hat{\theta}' = (C_2/2)\delta^{-2} \tag{3.8}$$

Equations (3.7) and (3.8) have solutions in terms of elliptic functions and, for a stable missile, δ varies periodically between a δ_{max} and a δ_{min}. Before considering this solution in detail, we first need to introduce quantities x_j that are similar to the squared modal amplitudes of the quasi-linear amplitude plane and that we will use to construct generalized amplitude planes. Therefore,

$$x_1^{1/2} + x_2^{1/2} = \delta_{max} \tag{3.9}$$

$$|x_1^{1/2} - x_2^{1/2}| = \delta_{min} \tag{3.10}$$

By use of Eqs. (3.9, 3.10, and 3.7), we can express our constants of the motion in terms of the generalized squared modal amplitudes

$$C_1 = -\hat{M}_0[2(x_1 + x_2) + (m_2/2)(3x_1^2 + 10x_1x_2 + 3x_2^2)] \tag{3.11}$$

$$C_2 = 2(x_1 - x_2)\ \{-\hat{M}_0[1 + m_2(x_1 + x_2)]\}^{1/2} \tag{3.12}$$

The actual form of elliptic function solution depends on the algebraic signs of the coefficients of the static-moment terms as modified by the spin.

Of the four possible combinations of signs, only three correspond to bounded motion and these may be classified in the following way (Fig. 1): a) stable at small angles, more stable at large angles ($\hat{M}_0 < 0$, $m_2 > 0$); b) stable at small angles, less stable at large angles ($\hat{M}_0 < 0$, $m_2 < 0$); and c) unstable at small angles, stable at large angles ($\hat{M}_0 > 0$, $m_2 < 0$). The solution of Eq. (3.4) for each of these cases is:[12]

Case a:

$$\delta^2 = x_1 + x_2 + 2(x_1x_2)^{1/2}[1 - 2sn^2(\omega s, k)] \tag{3.13}$$

Case b:

$$\delta^2 = x_1 + x_2 - 2(x_1x_2)^{1/2}[1 - 2sn^2(\tilde{\omega}s, k)] \qquad \omega^2 > 0 \tag{3.14}$$

Case c:

$$\begin{gathered}\delta^2 = x_1 + x_2 + 2(x_1x_2)^{1/2}[1 - 2sn^2(\omega s, k)] \\ \hat{M}_0[1 + m_2(x_1 + x_2)] \leqq 0\end{gathered} \tag{3.15}$$

where

$$\begin{aligned}\omega^2 &= -\hat{M}_0\{1 + (m_2/2)[3x_1 + 2(x_1x_2)^{1/2} + 3x_2]\} \\ \tilde{\omega}^2 &= -\hat{M}_0\{1 + (m_2/2)[3x_1 - 2(x_1x_2)^{1/2} + 3x_2]\} \\ k^2 &= -2\hat{M}_0 m_2\omega^{-2}(x_1x_2)^{1/2} \qquad \text{(types a and c)} \\ &= 2\hat{M}_0 m_2\tilde{\omega}^{-2}(x_1x_2)^{1/2} \qquad \text{(type b)}\end{aligned}$$

4. Perturbation Method

The coefficients in Eq. (2.11) may be functions of the independent variable for several reasons: varying air density or varying aerodynamic coefficients as a result of their dependence on Mach number or Reynold's number. For simplicity, we limit ourselves to a consideration of varying air density due to ascent or descent through the earth's atmosphere.

The actual density variation can be reasonably well approximated‡ by an exponential up to altitudes of 300,000 ft:

$$\rho = \rho_0 e^{-\sigma z} \tag{4.1}$$

where z is altitude in feet and $\sigma = (22{,}000\ \text{ft})^{-1}$. If $\psi(s)$ is the angle the flight path makes with respect to the downward pointing vertical, z can be related to our independent variable s by the equation

$$z = -\int_0^s l\cos\psi(s_1)ds_1 \tag{4.2}$$

Therefore,

$$\hat{M}_0' = \tilde{\sigma}\hat{M}_0/(1 - s_g) \tag{4.3}$$

$$m_2' = -m_2\tilde{\sigma}P^2/4\hat{M}_0 \tag{4.4}$$

where $\tilde{\sigma} = \rho'/\rho = \sigma l\cos\psi$.

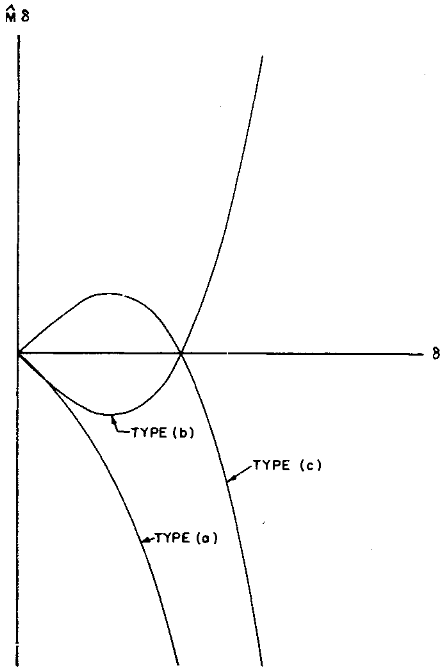

Fig. 1 Cubic static moments.

‡ The Air Research and Development Command standard atmosphere has been better approximated by a set of four exponentials that piecewise approximate the actual density variation.[13]

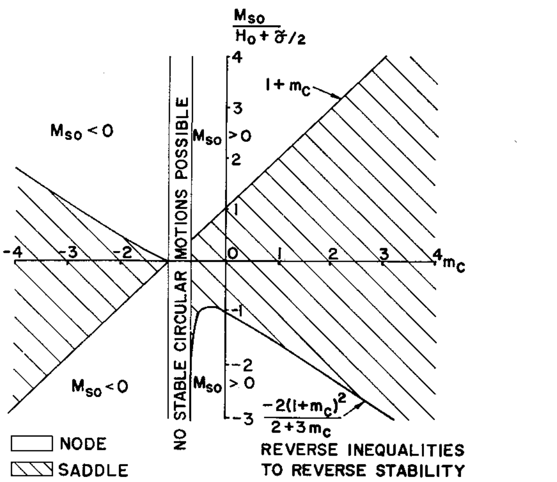

Fig. 2 Stable node for side moment $[M_s = M_{so}$ sign $(x_1 - x_2)]$; $H_2 > 0$.

For a cubic static moment, two differential equations for C_1 and C_2 were derived which showed these quantities to be constants of the motion. The same algebraic steps may be used on Eq. (2.11) to yield more general equations for C_1 and C_2, which now are not constants:

$$C_1' = -H[2C_1 + \hat{M}_0(2\delta^2 + m_2\delta^4)] + M^*(\delta^2)' + [M_s - PH/2]C_2 - \hat{M}_0\tilde{\sigma}\delta^2[(1 - s_g)^{-1} + (m_2/2)\delta^2] + \gamma'\gamma^{-1}[2C_1 + \hat{M}_0(2\delta^2 + m_2\delta^4) + PC_2/2] \quad (4.5)$$

$$C_2' = -HC_2 + [2M_s - PH]\delta^2 + \gamma'\gamma^{-1}[C_2 + P\delta^2] \quad (4.6)$$

If we assume the right sides of Eqs. (4.5) and (4.6) to be small, C_1 and C_2 are almost constants, and Eqs. (3.7 and 3.13–3.15) are good approximations of δ^2 and $(\delta^2)'$. Our basic assumption is that the essential effect of the nonlinearities in these quantities is small over a cycle of the nutation and that this effect can be computed by averaging the nonlinearities over a cycle of the motion. For most simple nonlinearities, the averaging process involves integrals of δ^{2n} which in turn means integrals of $sn^{2n}u$. These can be computed[12] in terms of complete elliptic integrals of the first and second kind and are functions of x_1 and x_2. Then Eqs. (3.11) and (3.12) may be used to construct generalized amplitude planes:

$$x_j' = \{[H + \tilde{\sigma}/2(1 - s_g)]a_{Hj}\}_a + [M^* a_{Mj}]_a + [(2M_s - PH)a_{sj}]_a + m_2\tilde{\sigma}P^2\ddot{a}_{Pj} \qquad (j = 1, 2) \quad (4.7)$$

where

$$a_{Hj} = \tfrac{1}{2}(\omega\tilde{\omega})^{-2}\hat{M}_0^2\{b_j[(C_1/\hat{M}_0) + \delta^2 + (m_2/2)\delta^4] + (-1)^j(b_j + 2m_2x_j)[1 + m_2(x_1 + x_2)](x_1 - x_2)\}$$

$$a_{Mj} = -(2\omega\tilde{\omega})^{-2}\hat{M}_0 b_j(\delta^2)'$$

$$a_{sj} = (2\omega\tilde{\omega})^{-2}\{-\hat{M}_0^3[1 + m_2(x_1 + x_2)]\}^{1/2} \times [b_j(x_2 - x_1) + (-1)^j(b_j + 2m_2x_j)\delta^2]$$

$$a_{Pj} = \tfrac{1}{8}(2\omega\tilde{\omega})^{-2}\hat{M}_0[b_j(3x_1^2 + 10x_1x_2 + 3x_2^2 - \delta_a^4) + -2(-1)^j(b_j + 2m_2x_j)(x_1^2 - x_2^2)]$$

$$b_1 = 1 + (m_2/2)(x_1 + 3x_2)$$

$$b_2 = 1 + (m_2/2)(3x_1 + x_2)$$

$$[\quad]_a = \frac{1}{2K}\int_0^{2K} [\quad] du$$

$u = \omega s$ or $\tilde{\omega} s$ and K is complete elliptic integral of first kind. These equations can be easily applied to the case of linear moments ($m_2 = M^* = 0$, $sn\ u = \sin u$, $K = \pi/2$, $\delta_a^2 = x_1 + x_2$). Therefore,

$$\frac{x_1'}{x_1} = \frac{(K_1^2)'}{K_1^2} = -\left[H + \frac{\tilde{\sigma}}{2(1 - s_g)}\right] - \left[M_s - \frac{PH}{2}\right][-\hat{M}_0]^{-1/2} \quad (4.8)$$

$$\frac{x_2'}{x_2} = \frac{(K_2^2)'}{(K_2^2)} = -\left[H + \frac{\tilde{\sigma}}{2(1 - s_g)}\right] + \left[M_s - \frac{PH}{2}\right][-\hat{M}_0]^{-1/2} \quad (4.9)$$

These are precisely the usual linear relations.[11]

In general, Eq. (4.7) involves algebraic combinations of complete elliptic integrals which can be computed numerically, but are difficult to deal with otherwise. For two special motions, specific relations are obtainable: almost circular motion and almost planar motion. After a consideration of these two special motions, a fair amount of information can be inferred about the complete generalized amplitude plane. In the next two sections we will study these special motions for simple examples, and then in the last section we will combine this information to give a fairly complete picture of all possible amplitude planes for a specific nonlinearity.

5. Almost Circular Motion

Even for a cubic static moment, the perturbation technique is greatly handicapped by the presence of complete elliptic integrals of the first and second kinds in δ_a^{2n}. The modulus of these integrals vanishes for circular motion, ($x_1 = 0$ or $x_2 = 0$) and can be expanded in terms of x_1 or x_2 for near circular motion. Another advantage of near circular motion is that any static moment can be represented by a cubic with coefficients evaluated for the amplitude of the motion, and thus our use of a cubic static moment is not a limitation on the actual static moment when near circular motions are considered. A third advantage of near circular motion rests on the experimental observation of "oval" or circular limit motions. The construction of nonlinear moments that cause stable circular limit motions should give some insight on these observed motions.

By means of well-known relations for elliptic functions [(127.01) and (900.00) of Ref. 12] for small k, a very simple approximation for the average value of a power of δ can be derived for almost circular motion ($x_1 \gg x_2$)

$$\delta_a^{2n} \doteq x_1^n + nx_1^{n-1}x_2\{n - [m_c/(2 + 3m_c)]\} \quad (5.1)$$

where $m_c = (m_2\delta^2_{\max})_c = m_2x_1$ and the subscript c denotes values for circular motion. C_1 also reduces to a simple expression that is linear in x_2:

$$C_1 = -(\hat{M}_0/2)[x_1(4 + 3m_c) + x_2(4 + 10m_c)] \quad (5.2)$$

The parameter m_c is of fundamental importance for this section and has a simple interpretation. It is the ratio of the nonlinear part of the static moment to the modified linear part and is positive if these parts are additive and negative if they oppose each other. Circular motions may be classified by their value of m_c into the three types of cubic moment. This can be done by the inequalities associated with Eqs. (3.13–3.15): type a: $0 < m_c < \infty$, type b: $-\frac{2}{3} < m_c < 0$, and type c: $-\infty < m_c \leqq -1$. The excluded interval $(-1, -\frac{2}{3})$ corresponds to a type b modified static moment and circular motions for which the modified static moment is negative but less in magnitude than $\frac{1}{3}$ its linear value. Although exactly circular motions exist for amplitudes in this range, any perturbation of these motions will grow and, hence, circular motions are not stable or physically possible.

To illustrate the perturbation method for almost circular motion, we will consider two simple cases for which the spin vanishes ($s_g = 0$) and the amplitude-dependent damping term is linear in δ^2 ($H = H_0 + H_2\delta^2$).

Example 5 .1: $M^* = 0$; $M_s = M_{s0}$ [sign($x_1 - x_2$)]

Although the linear value of the side moment M_s is zero for zero spin (Eq. 2.8), the nonlinear value is not necessarily zero. The linear symmetry argument rests on the fact that the only way the side moment can pick a direction is with reference to the spin and, hence, when spin is zero, there is no physically significant way to indicate the direction of the side moment and, hence, no side moment is possible. The pitching and yawing motion, however, usually has a velocity component perpendicular to the angle δ, and thus a side moment can be physically described as reinforcing or opposing this motion. The function M_s is defined previously as $-M_{s0}$ for $x_2 > x_1$ (clockwise motion), 0 for $x_1 = x_2$ (planar motion), and M_{s0} for $x_1 > x_2$ (counterclockwise motion). Thus, this nonlinear side moment is possible for a rotationally symmetric missile. A positive M_{s0} corresponds to a moment that increases the angular motion about the flight path. This moment could be produced by unsymmetric separated flow over a missile with an elliptical or circular angular motion.

For this example, Eq. (4.7) becomes

$$x_1'/x_1 = -2(\hat{M}_0/2\omega\tilde{\omega})^2(2 + 3m_c)\{(1 + m_c) \times [H_0 + (\tilde{\sigma}/2) + H_2x_1] - M_{s0}\} \quad (5.3)$$

$$x_2'/x_2 = -(\hat{M}_0/2\omega\tilde{\omega})^2[(4 + 9m_c + 6m_c^2)(H_0 + \tilde{\sigma}/2) - m_cH_2x_1 + (4 + 9m_c)M_{s0}] \quad (5.4)$$

According to Eq. (5.3), a stable circular singularity exists with squared amplitude

$$\delta^2 = x_1 = [(1 + m_c)^{-1}M_{s0} - H_0 - \tilde{\sigma}/2]H_2^{-1} \quad (5.5)$$

if

$$H_2 > 0 \quad (5.6)$$

$$(1 + m_c)^{-1} M_{s0} - H_0 - \tilde{\sigma}/2 > 0 \quad (5.7)$$

If almost circular motions in the vicinity of this singularity approach it, then it is a stable node in the generalized amplitude plane and represents a stable limit motion. If these almost circular motions depart from it, it is a saddle point. For a stable limit motion then,

$$\tfrac{1}{2}(2 + 3m_c)(1 + m_c)^{-2} M_{s0} + H_0 + \tilde{\sigma}/2 > 0 \quad (5.8)$$

These inequalities are summarized in Fig. 2. From this figure, we see that circular limit cycles are possible for a linear static moment if M_{s0} is large enough. This is true even when the damping moment is stabilizing for small amplitudes ($H_0 > 0$) and becomes more stabilizing at larger amplitudes ($H_2 > 0$)!

This example mathematically is a very simple special case of a nonlinear side moment for a nonspinning symmetric missile, but the moment is discontinuous for planar motion ($x_1 = x_2$). This objection can easily be removed by somewhat more complex expressions such as a side moment coefficient proportional to the angular momentum:

$$M_s = M_{sc_2}C_2 \quad (5.9)$$

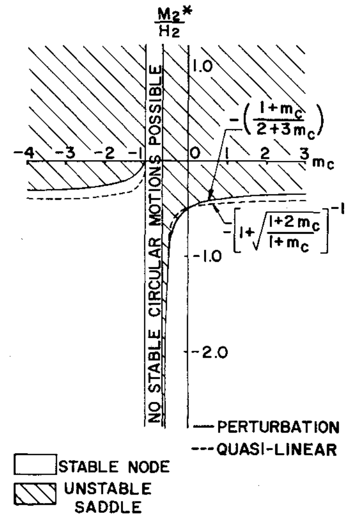

Fig. 3 Circular singularities for nonlinear damping moments ($H_0 + \tilde{\sigma}/2$) < 0; $H_2 > 0$.

Example 5.2: $M^* = M_2^*(\delta^2)'$; $M_s = 0$

As can be seen from Eq. (2.6), c^* appears in a damping moment term with the orientation of $-i\xi$, whereas d appears in a damping moment term with the orientation of $-i\xi'$. The relative direction of these two quantities changes from collinearity to normality as the motion changes from planar to circular.[14] In this example, the simplest nonlinear form of c^* and d is examined.

For this example, Eq. (4.7) becomes

$$x_1'/x_1 = -2(\hat{M}_0/2\omega\tilde{\omega})^2(1 + m_c)(2 + 3m_c) \times [H_0 + (\tilde{\sigma}/2) + H_2x_1] \quad (5.10)$$

$$x_2'/x_2 = -(\hat{M}_0/2\omega\tilde{\omega})^2\{(4 + 9m_c + 6m_c^2) \times [H_0 + (\tilde{\sigma}/2)] - [m_cH_2 + 2(2 + 3m_c)^2M_2^*]x_1\} \quad (5.11)$$

For these equations, a singularity is located at

$$m_c = m_2x_1 = -m_2(H_0 + \tilde{\sigma}/2)H_2^{-1}$$

and the conditions for a stable node are summarized in Fig. 3. Note that M^* must be nonzero for circular limit cycles to occur. The quasi-linear results of Ref. 8 are also shown for comparison.

6. Almost Planar Motion

As can be seen from Eq. (3.12), the angular momentum will be zero when $x_1 = x_2$ or $1 + m_2(x_1 + x_2) = 0$. When this occurs, the angular motion of a nonspinning missile is confined to a plane (planar motion). In the first case, the motion

Table 1 Coefficients for symmetric planar motion[a]

$a_{2p} = k_p^{-2}[1 - E_pK_p^{-1}]$	$a_{21} = (\frac{1}{2})k_1^2k_p^{-2}[-1 + (1 - k_p^2)^{-1}(E_p/K_p)^2]$
$a_{4p} = (\frac{1}{3})k_p^{-2}[2(1 + k_p^2)a_{2p} - 1]$	$a_{41} = (\frac{1}{3})k_p^{-2}[k_1^2(1 - 2a_{2p}) + 2(1 + k_p^2)a_{21}]$
$a_{6p} = (\frac{1}{5})k_p^{-2}[4(1 + k_p^2)a_{4p} - 3a_{2p}]$	$a_{61} = (\frac{1}{5})k_p^{-2}[k_1^2(3a_{2p} - 4a_{4p}) + 4(1 + k_p^2)a_{41} - 3a_{21}]$
$m_p < -2,\ m_p > 0 \qquad k_p^2 = (\frac{1}{2})m_p(1 + m_p)^{-1}$	$k_1^2 = m_p^{-1}k_p^2$
$A_2 = 1 - a_{2p}$	$B_2 = (\frac{1}{2})A_2 - a_{21}$
$A_4 = 1 - 2a_{2p} + a_{4p}$	$B_4 = A_4 - 2a_{21} + a_{41}$
$A_6 = 1 - 3a_{2p} + 3a_{4p} - a_{6p}$	$B_6 = (\frac{3}{2})A_6 - 3a_{21} + 3a_{41} - a_{61}$
$-1 < m_p < 0 \qquad k_p^2 = -m_p(2 + m_p)^{-1}$	$k_1^2 = -m_p^{-1}k_p^2$
$A_2 = a_{2p}$	$B_2 = (\frac{1}{2})A_2 + a_{21}$
$A_4 = a_{4p}$	$B_4 = A_4 + a_{41}$
$A_6 = a_{6p}$	$B_6 = (\frac{3}{2})A_6 + a_{61}$

[a] $K_p = K(k_p)$ and $E_p = E(k_p)$ are complete elliptic integrals of the first and second kinds.

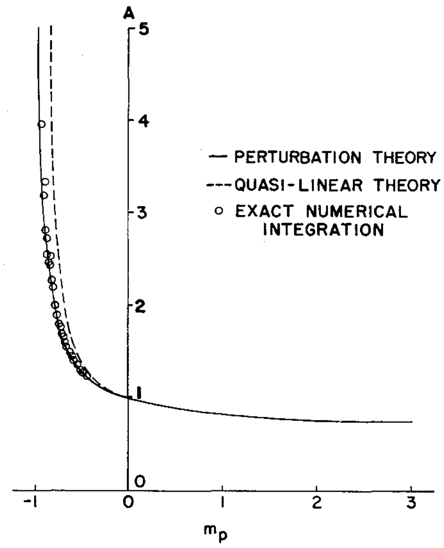

Fig. 4 ***A* vs m_p (types a and b moments).**

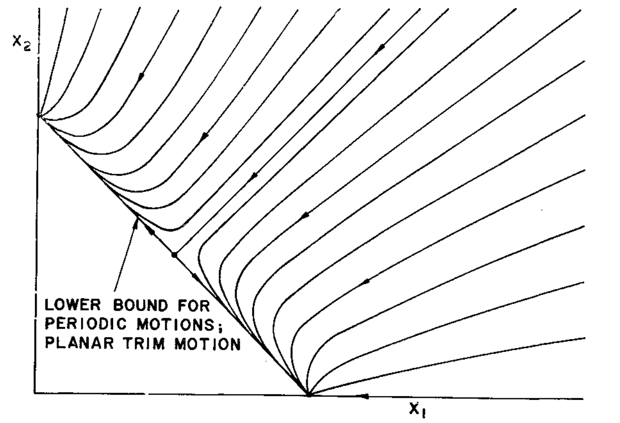

Fig. 6 Generalized amplitude plane for type c moment and linear damping.

varies from zero to a maximum angle of attack $\alpha_{\max} = 2x_1^{1/2} = 2x_2^{1/2}$ and is symmetric with respect to the origin. The second case can only occur for the type c moment and for $x_1 > x_2$ is a planar oscillation of amplitude $\alpha_{\max} = x_2^{1/2}$ about an average angle $\alpha_{av} = x_1^{1/2} = [-m_2^{-1} - x_2]^{1/2}$. For small amplitude oscillations, this average angle is reduced to the equilibrium trim angle of $(-m_2)^{-1/2}$.

The attractive properties of planar motion may be seen from Eq. (4.7). Planar motion of a nonspinning missile with no side moment will remain planar, whereas Eq. (4.7) reduces to essentially the same expression. This symmetry considerably simplifies the general problem of locating singularities.

The type of planar singularity can then be determined by an expansion about the singularity:

$$x_j = x_{j0}(1 + \epsilon_j) \tag{6.1}$$

where $x_{10}x_{20} \neq 0$, $x_{10} - x_{20} = 0$ for symmetric planar motion, and $x_{10} + x_{20} = -m_2^{-1}$ for planar trim motion. As can be seen from Eqs. (3.13–3.15), the range of variation of δ^2 is $4(x_1x_2)^{1/2}$. For symmetric motion this is $\alpha_{\max}^2$, whereas for motion about trim this is four times the product of $\alpha_{\max}$ and α_{av}. The average values of δ^{2n} can then be given as

$$\delta_a^{2n} = [A_{2n} + B_{2n}(\epsilon_1 + \epsilon_2)]\delta_p^{2n} \tag{6.2}$$

where A_{2n} and B_{2n} are functions of m_p and $m_p = 4m_2(x_1x_2)^{1/2}$. Some values of A_{2n} and B_{2n} are given in Table 1, but a more complete list is given in Ref. 15. m_p for symmetric planar motion is the ratio of the nonlinear part of the static moment to the linear part for maximum amplitude. From the inequalities associated with Eqs. (3.13–3.15), different intervals of m_p may be associated with the type of static moment: type a: $0 < m_p < \infty$, symmetric motion; type b: $-1 < m_p < 0$, symmetric motion; and type c: $-\infty < m_p < -2$, symmetric motion and $-2 < m_p < 0$, motion about trim. Note that the regions for symmetric motion do not overlap and so all three moment types may be considered by use of the complete range in m_p.

Example 6.1: $M_s = M^* = 0$; H = const

For this example of simple linear damping and cubic static moment, the damping of planar motion is given by the following equation:

$$\alpha'_{\max}/\alpha_{\max} = -[(H_0/2) + (\tilde{\sigma}/4)]A \tag{6.3}$$

where $A = (1 + m_p)^{-1}[2 + m_p - 2A_2 - m_pA_4]$ for symmetric motion, and $A = -m_p^{-2}[1 + 2(4 - m_p^2)^{-1/2}][(4 - m_p^2) + 8m_pA_2 + 4m_p^2A_4]$ for trim motion where A_2, A_4 are given in the table for symmetric motion, and, for trim motion,

$$A_2 = k_p^{-2}E_pK_p^{-1}$$

$$A_4 = \tfrac{1}{3}[k_p^{-2} - k_p^{-4} - 2A_2(1 - 2k_p^{-2})]$$

$$k_p^2 = -2m_p(2 - m_p)^{-1}$$

The quasi-linear value of A for symmetric planar motion is[7]

$$A = 2(4 + 3m_p)(8 + 9m_p)^{-1} \tag{6.4}$$

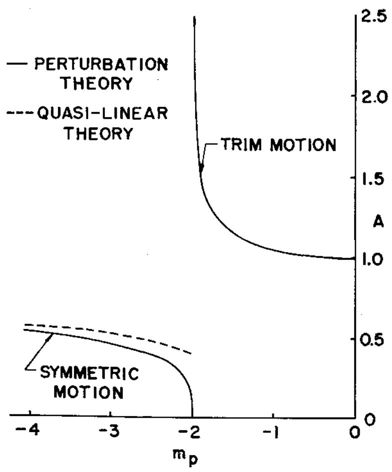

Fig. 5 ***A* vs m_p (type c moment).**

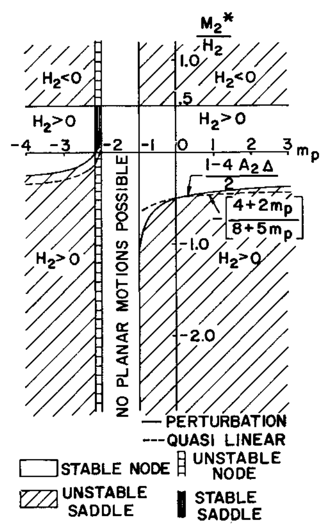

Fig. 7 Planar singularities $(H_0 + \tilde{\sigma}/2) < 0$; reverse signs of $(H_0 + \tilde{\sigma}/2)$ and H_2 to reverse stability.

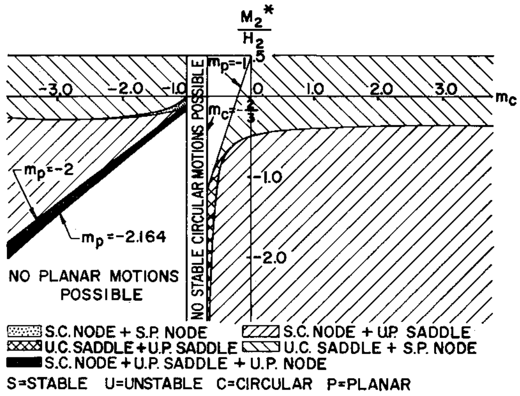

Fig. 8 Planar and circular singularities ($H_0 + \tilde{\sigma}/2$) < 0; $H_2 > 0$; reverse signs of ($H_0 + \tilde{\sigma}/2$) and H_2 to reverse stability.

These two values are compared for types a and b moments in Fig. 4 and for type c moment in Fig. 5. The actual 1–2% separation of the curves for type a cannot be seen, but the substantial separation near $m_p = -1$ is clear. Digital computation of the exact equations shows that the perturbation damping factor given by Eq. (6.3) is quite good.

The transition from symmetric motion to trim motion for the type c moment is quite interesting. The generalized amplitude plane for this case is shown in Fig. 6. Symmetric planar motion occurs on the line $x_1 - x_2 = 0$, whereas trim planar motion occurs on the line $x_1 + x_2 = -m_2^{-1}$. The intersection of these lines is a saddle, and the intercepts of the second line with axes are stable nodes.

Example 6.2: $P = M_s = 0$; $H = H_0 + H_2\delta^2$; $M^* = M_2^*(\delta^2)'$

For this case, the behavior of almost planar motions can be determined by the differential equations for the ϵ_j's of Eq. (6.1):

$$(2\tilde{\omega}\omega/M_0)^2\epsilon_1' = -(2 + m_p)[b_0 + b_1(\epsilon_1 + \epsilon_2) + (1 + m_p)b_2(\epsilon_1 - \epsilon_2)] - \tfrac{1}{4}m_p b_0(\epsilon_1 + 3\epsilon_2) \quad (6.5)$$

where

$$b_0 = 2\{(H_0 + \tilde{\sigma}/2)[2 + m_p - 2A_2 - m_pA_4] + [H_2 - 2M_2^*]\delta_p^2[(2 + m_p)A_2 - 2A_4 - m_pA_6]\}$$

$$b_1 = 2\{(H_0 + \tilde{\sigma}/2)[1 + m_p - 2B_2 - m_pB_4] + [H_2 - 2M_2^*]\delta_p^2[(1 + m_p)A_2 + (2 + m_p)B_2 - 2B_4 - m_pB_6]\}$$

$$b_2 = H_0 + \tilde{\sigma}/2 + H_2A_2\delta_p^2$$

and a similar equation applies for ϵ_2 with ϵ_1 and ϵ_2 interchanged. At a singular planar point b_0 vanishes or

$$m_p = m_2\delta_p^2 = -4m_2H_0(H_2 - 2M_2^*)^{-1}\Delta \quad (6.6)$$

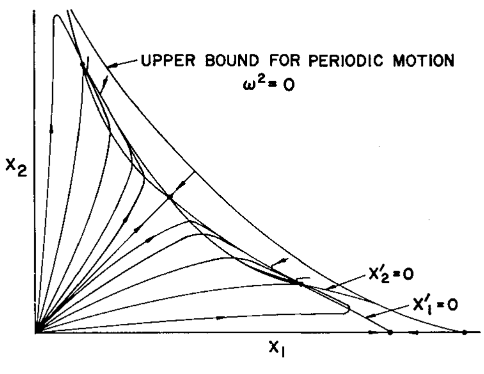

Fig. 9 Amplitude plane; type b moment ($m_c = -0.55$, $M_2^*/H_2 = -1$).

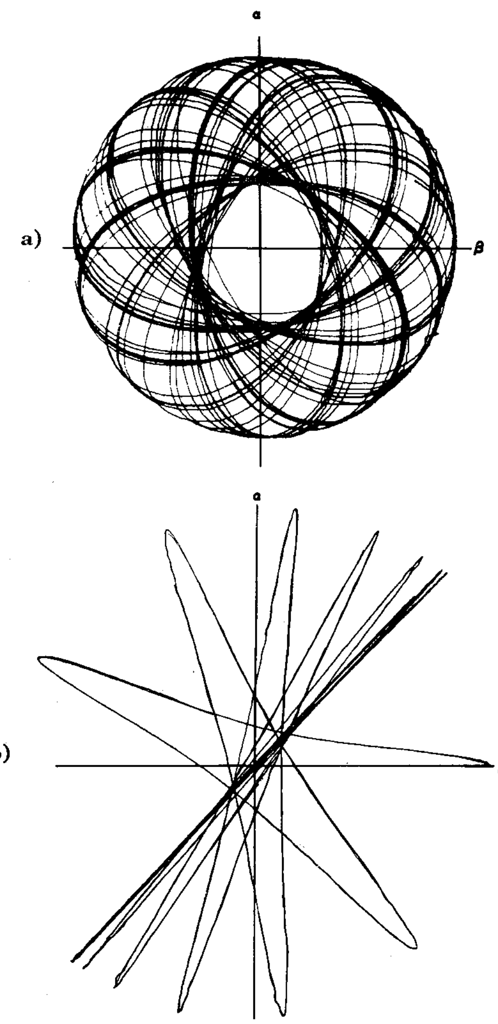

Fig. 10 Pitching and yawing limit motion; a) "oval" limit motion, type b moment (Fig. 9); b) planar limit motion, type c moment (Fig. 11).

where

$$\Delta = \frac{2 + m_p - 2A_2 - m_pA_4}{4[(2 + m_p)A_2 - 2A_4 - m_pA_6]}$$

The exact character of the singularities can be determined from the differential equations for ϵ_j and the usual criteria.[16] These criteria are presented in terms of H_0, H_2, M_2^*, and m_p in Fig. 7 and are compared with those based on the quasi-linear analysis.[8] (The unstable saddle of this figure is defined to be one for which motion on the origin side of the saddle is away from the origin.)

7. General Motions

Equations (4.7) are first-order nonlinear differential equations and, therefore, are much simpler than the fourth-order nonlinear differential equations of pitching and yawing motion. They always can be numerically integrated and the generalized amplitude planes generated. These integrations do, however, hide the effect of various parameters such as H_2 or M_2^*. The preceding treatments of the special cases of almost circular and almost planar motion can be combined to yield a fairly good idea of the generalized amplitude plane for different values of the parameters.

For the second example of Secs. 5 and 6, we can relate the m_c and m_p by

$$m_c/m_p = [1 - (2M_2^*/H_2)](1/4\Delta) \quad (7.1)$$

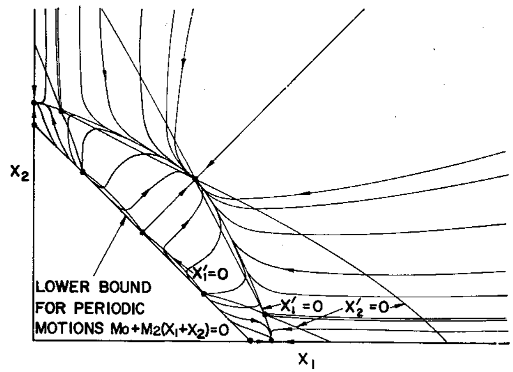

Fig. 11 Amplitude plane; type c moment (m_c, = −1.1, $M_2^*/H_2 = -\frac{1}{9}$).

This equation can now be used to replot Fig. 7 in terms of m_c, and the resulting figure can be superimposed on Fig. 3. The result is Fig. 8.

This figure has a number of interesting features. For a type b moment, it can be seen that as M_2^*/H_2 is increased, a circular node and planar saddle change to a circular saddle and a planar saddle that in turn change to a circular saddle and a planar node! This disappearance of the node from the x_1 axis and latter appearance on the line $x_1 = x_2$ implies its motion through the generalized amplitude plane and the possible existence of a limit motion varying between two nonzero amplitudes. This conjecture was checked by the numerical calculation of the generalized amplitude plane for $m_c = -0.55$ and $M_2^*/H_2 = -1$ (Fig. 9). This figure demonstrates the existence of the stable node. The differential equations of pitching and yawing motion for these parametric values were then calculated on an analog computer, and it was found that the motion quickly went to a limit motion with constant maximum and minimum amplitudes (Fig. 10). These maximum and minimum amplitudes are those assigned by the location of the node in the generalized amplitude plane.

Finally, we note that the dotted region of Fig. 8 appears to be quite interesting because of the presence of two stable limit motions. A sample generalized amplitude plane is shown in Fig. 11 and a plot of the angular motion approaching the planar motion is given in Fig. 10b. The ability of the perturbation method to predict fairly unexpected motions is very impressive.

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